Functional analysis
List 4

Zad 1. Show that a linear operator $A : X \to Y$ is bounded and calculate its norm.

N	X	Y	A
1.	l_3	$L_1[0,1]$	$(Ax)(t) = \sum_{k=1}^{\infty} \frac{x(k)t^k}{2^k}$
2.	$L_2[-1,1]$	l_1	$Ax = \left(\frac{1}{3}\int_{-1}^{1} tx(t)dt, ..., \frac{1}{3^k}\int_{-1}^{1} tx(t)dt, ...\right)$
3.	$C[-1,1]$	l_1	$Ax = \left(\frac{1}{3}\int_{-1}^{1} tx(t)dt, ..., \frac{1}{3^k}\int_{-1}^{1} tx(t)dt, ...\right)$
4.	$C^{(1)}[0,2]$	$C^{(1)}[0,1]$	$(Ax)(t) = tx(t^2+1)$
5.	$C[-17,13]$	$L_{\frac{5}{2}}[0,1]$	$(Ax)(t) = \int_0^1 (t+s)x(\sqrt{s})\, ds + 3x(0)$
6.	l_2	c	$(Ax)(t) = \left(\frac{x(1)}{1}, \frac{x(2)}{2}, \frac{x(3)}{3}, ...\right)$

Zad 2. Show that spaces c_0, c, l_p where $1 \leq p < \infty$ are seperable. Show that space l_∞ is not seperable.

Zad 3. Show the isometric isomorphism of the following spaces: $c_0^* \cong l_1$, $c^* \cong l_1$, $l_1^* \cong l_\infty$ and $l_p \cong l_q$ where $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Are c_0, c reflexive spaces?

Zad 4. Calculate the norm of functional $f : X \to \mathbb{C}$ given by

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} x(k), \qquad x = (x(1), x(2), ...) \in X,$$

where $X = c_0$, c, l_p dla $1 \leq p \leq \infty$.

Zad 5. Let $K : C[0,1] \to C[0,1]$. Show that an operator K is continuous and invertible. Find K^{-1} and the spectral radius $r(I - K)$ of $I - K$, if

a) $(Kx)(t) = x(t) + \frac{1}{5}\int_0^1 e^{t+s} x(s) ds$

b) $(Kx)(t) = x(t) - \int_0^1 s^2(t+1)x(s) ds$

Zad 6. Consider in space $C[0,1]$ an equation of the form

$$(A - \lambda 1)x = y, \qquad x, y \in C[0,1].$$

Find all parameters λ such that this equation possess a unique solution x for every given y, if

a) $(Ax)(t) = \int_0^1 e^{t+s} x(s) ds$

b) $(Ax)(t) = \int_0^1 (s^2 t + 1)x(s) ds$

c) $(Ax)(t) = \int_0^1 (s^2 t + t^2 s)x(s) ds$

d) $(Ax)(t) = \int_0^1 (st - e^t)x(s) ds$

What is the spectrum $\sigma(A)$ and the spectral radius $r(A)$ of an operator A?